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Star products and quantization of Poisson–Lie groups

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We prove some theorems by Drinfeld about solutions of the triangular quantum Yang–Baxter equation and corresponding quantum groups. These theorems are to be understood in the natural setting of invariant star products on a Lie group. We also set out and prove another theorem about the invariant Hochschild cohomological meaning of the quantum Yang–Baxter equation, which underlies the others.

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1. Introduction and results

1. A Poisson–Lie group $(\mathbf{G};\Lambda)$ is a Lie group endowed with a Poisson structure defined by a two-contravariant antisymmetric tensor Λ on $\mathbf{G}$ such that multiplication in $\mathbf{G}$ is a Poisson morphism [1]. If $\{\ ;\ \}$ means the Poisson bracket of $(\mathbf{G};\Lambda)$ or of $(\mathbf{G}\times\mathbf{G};\Lambda')$ [2,5,14], and if

$$\Delta : C^\infty(\mathbf{G}) \longrightarrow C^\infty(\mathbf{G})\,\hat{\otimes}\, C^\infty(\mathbf{G})$$

means the coproduct of the usual Hopf algebra $C^\infty(\mathbf{G})$, then $(\mathbf{G};\Lambda)$ is a Poisson–Lie group if and only if Δ is a Poisson morphism:

$$\Delta\{\varphi;\psi\} = \{\Delta\varphi;\Delta\psi\}, \quad \varphi,\psi\in C^\infty(\mathbf{G}). \tag{1}$$

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This notion has been considered by Drinfeld following, in particular, work by Faddeev and Sklyanin on the formulation of the *Inverse Scattering Method*, by means of classical and quantum Yang–Baxter equations. It is known [1] (see refs. [6,7,22]) that a simply connected Poisson–Lie group determines and is determined by a Lie bialgebra $(\mathfrak{g};\epsilon)$, where ϵ is a one-cocycle on $\mathfrak{g}$, with values on $\Lambda^2(\mathfrak{g})$, relatively to the adjoint representation and such that $\epsilon^t : \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ defines a Lie algebra structure on $\mathfrak{g}^*$.

An important case of Poisson–Lie groups is when ϵ is exact, i.e., $\epsilon = \delta r$, $r \in \mathfrak{g} \wedge \mathfrak{g}$.

Let $\mathfrak{A}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and define three elements:

$$r^{12} = r \otimes 1, \quad r^{13} = \mathbf{P}^{23} r^{12} \mathbf{P}^{23}, \quad r^{23} = 1 \otimes r \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g},$$

where $\mathbf{P}^{23}$ means the permutation (2, 3). We write

$$[r;r] \equiv [r^{12};r^{13}] + [r^{12};r^{23}] + [r^{13};r^{23}]$$

[calculations in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$]. If $(\mathbf{G};\Lambda)$ is a Poisson–Lie group determined by the bialgebra $(\mathfrak{g};\epsilon = \delta r)$ then r satisfies the generalized classical Yang–Baxter equation, i.e.,

$$\mathbf{ad}\, x \cdot [r;r] = 0, \quad \forall x \in \mathfrak{g}. \tag{2}$$

Conversely, if r satisfies (2), $(\mathfrak{g};\epsilon = \delta r)$ is the bialgebra of the simply connected Poisson–Lie group $(\mathbf{G};\Lambda)$.

Suppose r satisfies (2). Let

$$g \in \mathbf{G}, \quad \Lambda^\ell(g) = T_e L_g \cdot r, \quad \Lambda^{\mathrm{r}}(g) = T_e R_g \cdot r$$

be left- and right-invariant skew-symmetric two-tensors defined from r by left and right translation, respectively. The tensor Λ of the corresponding simply connected Poisson–Lie group $(\mathfrak{g};\epsilon = \delta r)$ is then

$$\Lambda = \Lambda^\ell - \Lambda^{\mathrm{r}}.$$

A subcase of the preceding is when r satisfies the classical Yang–Baxter equation:

$$[r;r] = 0.$$

The bialgebra and corresponding Poisson–Lie group are in this case called triangular. Λ^ℓ and Λ^{r} separately define invariant Poisson strucures on $\mathbf{G}$ (but not Poisson–Lie structures!), such that $\Lambda = \Lambda^\ell - \Lambda^{\mathrm{r}}$ is the Poisson–Lie structure on $\mathbf{G}$.

By our understanding of some of Drinfeld's work, the quantization of a triangular Poisson–Lie group could be the starting point for a general theory of quantum groups.

2. The notion of star product was systematically set out in refs. [8,11]. See refs. [12,13] for the theorem of existence of star products on arbitrary symplectic manifolds.

Let $(\mathbf{M}; \Lambda)$ be a Poisson manifold, and let $C^\infty(\mathbf{M})[[h]]$ be the algebra of formal power series in h with coefficients in $C^\infty(\mathbf{M})$. A star product is a bilinear mapping

$$C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M}) \xrightarrow{*} C^\infty(\mathbf{M})[[h]]$$

such that

$$\varphi * \psi = \varphi \cdot \psi + \sum_{i=1}^{\infty} C_i(\varphi; \psi)\, h^i, \tag{i}$$

where C_i is a bidifferential operator on $C^\infty(\mathbf{M})$, with no constant term on each argument, that is, $C_i(1; \psi) = C_i(\varphi; 1) = 0$, and hence $\varphi * 1 = \varphi$, $1 * \psi = \psi$;

$$(\varphi * \psi) * \chi = \varphi * (\psi * \chi), \tag{ii}$$

$$\frac{\varphi * \psi - \psi * \varphi}{h} = \{\varphi; \psi\} + O(\varphi; \psi; h). \tag{iii}$$

A star product is thus an associative deformation [9–11] of the usual algebra of functions in $C^\infty(\mathbf{M})$ where the two-cochains C_i are bidifferential operators as above. The natural setting for the development of star products theory is the *null-on-the-constants differential Hochschild cohomology*. See refs. [10,11,13] for some results on this cohomology which we will need for this work.

Clearly, a star product is linearly defined on $C^\infty(\mathbf{M})[[h]]$.

Topological considerations aside, a Poisson–Lie group $(\mathbf{G}; \Lambda)$ can be said to determine and be determined by its usual commutative and non-co-commutative Hopf algebra $(C^\infty(\mathbf{G}); \cdot\,; \Delta)$ satisfying (1).

To quantize $(\mathbf{G}; \Lambda)$ we should first endow $C^\infty(\mathbf{G})[[h]]$ with a non-commutative, non-co-commutative Hopf algebra structure, where the coproduct Δ is the same as that of $C^\infty(\mathbf{G})$ and the product, $*$, is a star product. When topological considerations are set aside, a quantum group can be defined as this Hopf algebra. See refs. [24–26], where in particular deformations of C^*-algebras are considered.

3. The problem is thus to get $*$ products on $(\mathbf{G}; \Lambda)$ such that the compatibility relation

$$\Delta(\varphi * \psi) = \Delta\varphi * \Delta\psi \tag{3}$$

is satisfied. The star product on the right-hand side is canonically defined on $C^\infty(\mathbf{G}) \hat{\otimes} C^\infty(\mathbf{G}) \equiv C^\infty(\mathbf{G} \times \mathbf{G})$.

In this work we consider the case of a triangular Poisson–Lie group $(\mathbf{G}; \Lambda)$ defined by a solution $r \in \Lambda^2(\mathfrak{g})$ of the classical Yang–Baxter equation $[r; r] = 0$.

Let $(\mathbf{G};\Lambda^\ell)$ be the left-invariant Poisson structure on $\mathbf{G}$ defined by r, i.e., $\Lambda^\ell = T_eL\cdot r$. Let $*^\ell$ be a left-invariant star product on $\mathbf{G}$. In particular we have

$$\frac{\varphi *^\ell \psi - \psi *^\ell \varphi}{h} = \{\varphi;\psi\}^\ell + \mathcal{O}(\varphi;\psi;h).$$

As we will see below, $*^\ell$ is defined by an element

$$F(x;y) = 1 + \sum_{i=1}^{\infty} F_i(x;y)\, h^i$$

in $\mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})[[h]]$ such that

$$F(x+y;z)\,F(x;y) = F(x;y+z)\,F(y;z), \tag{4}$$

where $+$ means the action of the usual coproduct of $\mathfrak{A}(\mathfrak{g})$.

This is a form under which Drinfeld considered star products in his theory of quantum groups or more specifically in the theory of the Quantum Yang–Baxter Equation (QYBE) with no spectral parameter [15].

Let

$$C_i(\varphi;\psi) = (F_i(x;y))^\ell(\varphi\otimes\psi),$$

where $(F_i(x;y))^\ell$ is the left-invariant bidifferential operator on $\mathbf{G}$, determined by $F_i(x;y)\in\mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})$. Left invariance of $*^\ell$, or of $(F_i(x;y))^\ell$ means that

$$L_g(\varphi * \psi) = L_g\varphi * L_g\psi, \quad \forall g\in\mathbf{G}$$

[but not yet (3)!], where $(L_g\varphi)(g') = \varphi(g\cdot g')$. Right translations define similar objects, $*^{\mathrm{r}}$and $\Lambda^{\mathrm{r}} = T_eR\cdot r$. There is thus a unique element $H(x;y)\in\mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})[[h]]$ satisfying

$$H(x;y)\,H(x+y;z) = H(y;z)\,H(x;y+z), \tag{5}$$

and such that two-cochains, C_i, in $*^{\mathrm{r}}$ are defined by

$$C_i(\varphi;\psi) = (H_i(x;y))^{\mathrm{r}}(\varphi\otimes\psi). \tag{6}$$

In particular, if $F(x;y)$ satisfies (4), $F^{-1}(x;y)$ satisfies (5) and, by (6), defines a right-invariant star product on $\mathbf{G}$.

On the basis of work by Drinfeld, Takhtajan [23] considers the following expression:

$$\varphi \overset{\circ}{*} \psi = (F^{-1}(x;y))^{\mathrm{r}}(F(x;y))^\ell(\varphi\otimes\psi).$$

We will here prove that $\overset{\circ}{*}$ is a (definitely non-invariant) star product on $\mathbf{G}$ which satisfies (3).

We remark that, if in $(\mathbf{G};\Lambda)$ $\mathbf{G}$ is abelian, then $\Lambda = 0$ and $\varphi \overset{\circ}{*} \psi = \varphi\cdot\psi$.

Drinfeld obtains a $*^\ell$-product on $(\mathbf{G};\beta_1)$ where β_1 is now an invariant symplectic structure [which suffices to obtain $*$-products on triangular Poisson–Lie groups $(\mathbf{G};\Lambda)$ or $*^\ell$-products on a Poisson group $(\mathbf{G};\Lambda^\ell)$] by straightforward generalization of one way of getting the usual Moyal $*$-product on $(\mathbb{R}^{2n};\beta_1)$.

Specifically taking $\bar{\mathfrak{g}} = \mathfrak{g} \times_{\beta_1} \mathbb{R}$ as the central extension of $\mathfrak{g}$ by the two-cocycle β_1, Drinfeld defines an integral, containing the Campbell–Hausdorff groups of $\mathfrak{g}$ and $\bar{\mathfrak{g}}$, on an orbit of the coadjoint representation of the simply connected Lie group $\overline{\mathbf{G}}$. In the case $\mathbf{G}= \mathbb{R}^{2n}$, this integral is exactly the integral expression of the Moyal $*$-product in the p and q coordinates [3,4]. See ref. [16], where the authors clarify this construction. See refs. [20,21] for a detailed development.

Another theorem by Drinfeld can be stated as follows (see refs. [15,20]):

Any $*^{\ell}$-product on $(\mathbf{G};\beta_1)$ is equivalent to one obtained in the foregoing construction by considering the central extension of $\bar{\mathfrak{g}}_h = \mathfrak{g} \times_{\beta_h} \mathbb{R}$ with a de Rham invariant two-cocycle β_h on $\mathbf{G}$ of the form

$$\beta_h = \beta_1 + h\beta_2 + \cdots + h^{k-1}\beta_k ,$$

where β_i, $i \geq 2$, are any invariant cocycles and k is any natural number. Again, the notion of equivalent extensions allow us to choose β_i, $i \geq 2$, in some fixed supplementary space of the space of exact two-cocycles. To prove this theorem we require in particular:

(i) the isomorphism between the cohomology defined by the Schouten bracket of $(\mathbf{G};\beta_1)$ and the de Rahm cohomology as stated in ref. [19];

(ii) theorem 2 in this article, which states the cohomological meaning of the QYBE.

4. The relation between star products and the QYBE is as follows. Let $F(x;y)$ define a $*^{\ell}$-product on $\mathbf{G}$ (and hereafter designate this invariant $*^{\ell}$-product). Define, as Drinfeld does,

$$S(x;y) = F^{-1}(y;x)\,F(x;y)\,; \tag{7}$$

we then have

$$S(x;y)\,S(x;z)\,S(y;z) = S(y;z)\,S(x;z)\,S(x;y)\,,$$
$$S(x;y)\,S(y;x) = 1\,,$$

that is to say, $S(x;y)$ satisfies the triangular QYBE on $\mathfrak{A}(\mathfrak{g})[[h]]$.

We prove this theorem by Drinfeld below. See also refs. [15–17]. Conversely, we here prove the following theorem by Drinfeld (see refs. [15,18,17]):

Theorem 1. *Let* $r \in \mathrm{End}(\mathbb{R}^n \times \mathbb{R}^n)[[h]]$ *satisfy*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12},$$

$$R^{12}R^{21} = 1.$$

Then:

(i) There is a $*^{\ell}$*-product* $F(x;y)$ *on the Lie group* $\mathrm{GL}(n;\mathbb{R})$ *such that, if* $S(x;y)$ *is as in (7), we have*

$$(\mathbf{P}\otimes\mathbf{P})S = R, \tag{8}$$

where

$$\mathbf{P} : \mathfrak{gl}(n;\mathbb{R}) \longrightarrow \mathrm{End}(\mathbb{R}^n)$$

is the natural representation of the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$.

(ii) Any other $*^{\ell}$*-product* $F'(x;y)$ *which satisfies (8) is equivalent to* $F(x;y)$. *That is to say, there is an*

$$E(x) = 1 + \sum_{i=1}^{\infty} E_i(x)\, h^i \in \mathfrak{A}(\mathfrak{g})[[h]],$$

such that

$$F'(x;y) = E^{-1}(x+y)\, F(x;y)\, E(x)\, E(y).$$

Moreover, $E(x)$ *can be chosen so that* $\mathbf{P}E = 1$.

To prove this theorem, we first need to prove theorem 2 below.

If $\mathbf{d}$ means the differential in the invariant Hochschild complex on $\mathbf{G}$, $\mathcal{T}\mathfrak{A}(\mathfrak{g})$, relation (4) is equivalent to the set of relations

$$\mathbf{d}F_l(x;y;z) = \alpha_l(x;y;z), \quad l = 1,2,3,\ldots, \tag{9}$$

where

$$\alpha_l(x;y;z) = \sum_{\substack{i+j=l \\ i,j\geq 1}} [F_i(x+y;z)\, F_j(x;y) - F_i(x;y+z)\, F_j(y;z)].$$

Suppose now that these relations are satisfied with $l = 1,2,\ldots,k-1$. Then Gerstenhaber's theory [9,11] states that $\alpha_k(x;y;z)$ is a three-cocycle in the foregoing cohomology. But, from a theorem by Vey and Lichnerowicz we have

$$\alpha_k(x;y;z) = \mathbf{A}\alpha_k(x;y;z) + \mathbf{d}E_k(x;y;z), \tag{10}$$

where $\mathbf{A}\alpha_k(x;y;z)$ is the skew-symmetrical part of $\alpha_k(x;y;z)$ (which is a three-tensor on $\mathbf{G}$), and where $E_k(x;y)$ is a two-cochain. Thus (9) is also satisfied with $l = k$ if and only if $\alpha_k(x;y;z)$ is exact. With these notations (see section 7 below), we prove the following [18,20].

Theorem 2. *Let*

$$F(x;y) = 1 + \sum_{i=1}^{\infty} F_i(x;y)\, h^i$$

be an arbitrary element of $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$ *and* $S(x;y) = F^{-1}(y;x)F(x;y)$. *Suppose* $F(x;y)$ *is a star product to the order* $k-1$, *and hence satisfies (9) with* $l = 1,2,\ldots,k-1$. *Then in the Vey–Lichnerowicz splitting (10), we have*

$$\mathbf{A}\alpha_k(x;y;z) = -\tfrac{1}{6}[S(x;y)\, S(x;z)\, S(y;z) - S(y;z)\, S(x;z)\, S(x;y)]_k,$$

where the right-hand side means the coefficient of h^k *in the formal series defining the bracketed term.*

Definition 3. We call the relation

$$[S(x;y)\,S(x;z)\,S(y;z) - S(y;z)\,S(x;z)\,S(x;y)]_k = 0 \qquad (11)$$

the QYBE to order k.

Corollary 4. *The star product to order $k-1$ in theorem 2 can be extended to a star product to order k if and only if the corresponding QYBE to order k is satisfied.*

5. We end this Introduction with some words about the originality of our contribution. In our knowledge, the proofs given here do not appear in the literature, except for the relativily easy theorems 9, 18, see refs. [16,23]. In our understanding, theorem 2 is basic to Drinfeld's work [15], but it is not stated there and no reference is given to results of the theory of star products or, what at this point is the same, to results on invariant, differential, Hochschild cohomology on a Lie group. We believe, nevertheless, that this theorem was deeply understood by the author of ref. [15], when this reference was written.

6. An important theorem in the theory of star products was proved for the first time by M. De Wilde and P. Lecomte in ref. [12]. It states that *on an arbitrary symplectic manifold there exists a star product.* See ref. [13], where additional results are also obtained.

2. The triangular quantum Yang–Baxter equation

1. Let $\mathbf{V}$ be a real vector space and R an element in $\mathrm{End}(\mathbf{V}\otimes\mathbf{V})$. Let us define the following operators:

$$R^{12} \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}), \qquad R^{12} = R\otimes I,$$

$$R^{13} \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}), \qquad R^{23} = I\otimes R,$$

$$R^{13} \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}), \qquad R^{13} = P^{23}R^{12}P^{23}.$$

The triangular Quantum Yang–Baxter Equation (QYBE) with no spectral parameter is by definition the system of equations

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \qquad \text{(i)}$$

$$R^{12}R^{21} = I, \quad R^{21} = P^{12}R^{12}P^{12}. \qquad \text{(ii)}$$

2. Drinfeld's idea was to look for solutions of (i) and (ii) in the space of formal power series in h with coefficients in End$(\mathbf{V}\otimes\mathbf{V})$. If

$$R = I + \sum_{i=1}^{\infty} r_i h^i, \quad r_i \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}),$$

eqs. (i), (ii) are to order h^2 and h^1, respectively,

$$r_1^{12}r_1^{13} + r_1^{12}r_1^{23} + r_1^{13}r_1^{23} = r_1^{23}r_1^{13} + r_1^{23}r_1^{12} + r_1^{13}r_1^{12}, \qquad \text{(iii)}$$

$$r_1^{12} + r_1^{21} = 0; \qquad \text{(iv)}$$

that is to say,

$$[r_1^{12};r_1^{13}] + [r_1^{12};r_1^{23}] + [r_1^{13};r_1^{23}] = 0, \qquad \text{(v)}$$

$$\text{(vi)} \equiv \text{(iv)}. \qquad \text{(vi)}$$

Clearly, these equations have a meaning on any Lie algebra $\mathfrak{g}$ and not just on $\mathfrak{gl}(n;\mathbb{R})$. If products are considered in the enveloping algebra $\mathfrak{A}(\mathfrak{g})$, and if we put $S_1 \in \mathfrak{g}\wedge\mathfrak{g}$ in place of r_1, we can write

$$S_1^{12}S_1^{13} + S_1^{12}S_1^{23} + S_1^{13}S_1^{23} = S_1^{23}S_1^{13} + S_1^{23}S_1^{12} + S_1^{13}S_1^{12}, \qquad \text{(iii}')$$

$$S_1^{12} + S_1^{21} = 0. \qquad \text{(iv}')$$

Then we look for

$$S \in \mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})[[h]], \quad S = I + \sum_{i=1}^{\infty} S_i h^i,$$

such that the QYBE is satified in $\mathfrak{A}(\mathfrak{g})^{\otimes 3}$,

$$S^{12}S^{13}S^{23} = S^{23}S^{13}S^{12}, \qquad \text{(i}')$$

$$S^{12}S^{21} = I. \qquad \text{(ii}')$$

Equations (i) and (ii) are now obtained by considering some representation $\pi : \mathfrak{A}(\mathfrak{g}) \longrightarrow \mathrm{End}(\mathbf{V})$ and by defining $R = (\pi\otimes\pi)S$. The problem is then to solve (i′) and (ii′). The theory of invariant star products on a Lie group provides that solution.

3. Invariant differential operators on G

1. Let $T_e\mathbf{G}$ be the vector space tangent to $\mathbf{G}$ at the unit e of $\mathbf{G}$. If $x \in T_e\mathbf{G}$, then X^ℓ and X^r are vector fields on $\mathbf{G}$ generated from x by left and right translations,

$$X^\ell(g) = T_eL_g\cdot x, \quad X^r = T_eR_g\cdot x, \quad \forall g \in \mathbf{G};$$

if $y \in T_e\mathbf{G}$, there is $[x;y] \in T_e\mathbf{G}$ such that

$$[X^\ell; Y^\ell](g) = T_e L_g \cdot [x;y],$$

$$[X^r; Y^r](g) = -T_e R_g \cdot [x;y].$$

The Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ is the vector space $T_e\mathbf{G}$ endowed with the bracket[;].

Let $\mathcal{D}^\ell(\mathbf{G})$ be the associative algebra generated by vector fields X^ℓ, $x \in T_e\mathbf{G}$. This is the algebra of left-invariant differential operators on $\mathbf{G}$. In the same way, let $\mathcal{D}^r(\mathbf{G})$ be the algebra of right-invariant differential operators. Let

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g} \otimes \overset{k}{\cdots} \otimes \mathfrak{g}$$

be the tensor algebra on the vector space $\mathfrak{g}$, and $\mathcal{J}$ the bilateral ideal generated by the relations $x \otimes y - y \otimes x - [x;y]$. The enveloping algebra $\mathfrak{A}(\mathfrak{g})$ is by definition the associative algebra $\mathfrak{A}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{J}$. The mapping $x \longrightarrow X^\ell$ extends to an algebra isomorphism from $\mathfrak{A}(\mathfrak{g})$ to $\mathcal{D}^\ell(\mathbf{G})$. In the same way, if $\mathfrak{A}(\mathfrak{g})^\circ$ is the opposed algebra to $\mathfrak{A}(\mathfrak{g})$, the mapping $x \longrightarrow X^r$ extends to an algebra isomorphism from $\mathfrak{A}(\mathfrak{g})^\circ$ to $\mathcal{D}^r(\mathbf{G})$. Thus if $x \cdot y \cdots z$ is a product in $\mathfrak{A}(\mathfrak{g})$, we then have

$$(x \cdot y \cdots z)^\ell = x^\ell \cdot y^\ell \cdots z^\ell,$$

$$(x \cdot y \cdots z)^r = z^r \cdots y^r \cdot x^r.$$

The mapping

$$\mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}, \quad x \longrightarrow x \otimes 1 + 1 \otimes x,$$

is linear. It extends in a unique way to a homomorphism of algebras,

$$\mathbf{c} : \mathfrak{A}(\mathfrak{g}) \longrightarrow \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g}).$$

This mapping $\mathbf{c}$ is the coproduct of $\mathfrak{A}(\mathfrak{g})$.

Remark. Let φ, ψ be two elements in $C^\infty(\mathbf{G})$, and $x \in \mathfrak{g}$; then

$$X^\ell(\varphi \cdot \psi) = (\mathbf{c}(x))^\ell(\varphi \otimes \psi),$$

$$X^r(\varphi \cdot \psi) = (\mathbf{c}(x))^r(\varphi \otimes \psi).$$

It will be convenient to introduce the following usual polynomial notation for elements in $\mathfrak{A}(\mathfrak{g})$. If $\{e_i,\ i = 1, \dots, n\}$ is a basis of $\mathfrak{g}$, we will write

$$x_i = e_i \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots \quad (i = 1, 2, \dots, n),$$

whatever the number of 1's. Thus

$$x_i \in \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g}) \otimes \cdots.$$

We will also write

$$y_i = 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots, \quad y_i \in \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g}) \otimes \cdots,$$

and then

$$z_i = 1 \otimes 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots,$$

$$t_i = 1 \otimes 1 \otimes 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1 \otimes \cdots,$$

etc. Elements x commute with elements y, but if $i \neq j$, x_i and x_j are in general non-commuting. The same is true for $y, z, \ldots$ etc. An element $A \in \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$ can then be written as a polynomial in the variables x, y,

$$A = \sum a_{(\mu)(\nu)} e_1^{\mu_1} \cdots e_n^{\mu_n} \otimes e_1^{\nu_1} \cdots e_n^{\nu_n}$$

$$= \sum a_{(\mu)(\nu)} x_1^{\mu_1} \cdots x_n^{\mu_n} \cdot y_1^{\nu_1} \cdots y_n^{\nu_n}.$$

We can write

$$\mathrm{c}(x_i) = (e_i \otimes 1 + 1 \otimes e_i) \otimes 1 \otimes \cdots = x_i + y_i,$$

so

$$\mathrm{c}(x_1^{\mu_1} \cdots x_n^{\mu_n}) = (x_1 + y_1)^{\mu_1} \cdots (x_n + y_n)^{\mu_n}.$$

If $P(x) \in \mathfrak{A}(\mathfrak{g})$, we consequently have $\mathrm{c}(P(x)) = P(x + y)$.

2. Let A be an element in $\mathfrak{A}(\mathfrak{g})$, and $A^\ell \in \mathcal{D}^\ell(\mathbf{G})$, $A^\mathrm{r} \in \mathcal{D}^\mathrm{r}(\mathbf{G})$, the corresponding invariant differential operators. The invariance properties are expressed as

$$(A^\ell f) \circ L_{g_1} = A^\ell (f \circ L_{g_1}), \qquad (A^\mathrm{r} f) \circ R_{g_1} = A^\mathrm{r}(f \circ R_{g_1}),$$

$$\forall g_1 \in \mathbf{G}, \quad f \in C^\infty(\mathbf{G}).$$

In a more convenient notation, we will write

$$A^\ell(g_1 \cdot g_2) f(g_1 \cdot g_2) = A^\ell(g_2) f(g_1 \cdot g_2),$$

$$A^\mathrm{r}(g_1 \cdot g_2) f(g_1 \cdot g_2) = A^\mathrm{r}(g_1) f(g_1 \cdot g_2).$$

Lemma 5. *Let $x_1, \cdots x_n$ be elements in* $\mathfrak{g}$. *Then*

$$(x_1 \cdots x_n)^\ell(g_1)\,(f \circ R_{g_2})(g_1) = (x_1 \cdots x_n)^\mathrm{r}(g_2)\,(f \circ L_{g_1})(g_2).$$

Proof.

$$(x_1 \cdots x_n)^\ell(g_1) f(g_1 \cdot g_2) = x_1^\ell(g_1) \cdots x_n^\ell(g_1) f(g_1 \cdot g_2)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t_1} \cdots \frac{\mathrm{d}}{\mathrm{d}t_n} f(g_1 \cdot \exp t_1 x_1 \cdot \exp t_2 x_2 \cdots \exp t_n x_n \cdot g_2)|_{t_1 = \cdots = t_n = 0}$$

$$= x_n^\mathrm{r}(g_2) \cdots x_1^\mathrm{r}(g_1) f(g_1 \cdot g_2) = (x_1 \cdots x_n)^\mathrm{r}(g_2) f(g_1 \cdot g_2). \qquad \square$$

Let

$$B(x;y) = \sum_{i=1}^{\infty} B_i(x;y)\, h^i, \quad B_0(x;y) = 1,$$

be an element of $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$ and B_i^{ℓ} , B_i^{r} the corresponding invariant bidifferential operators. We will write

$$B^{\ell}(g;g) = \sum_{i=0}^{\infty} B_i^{\ell}(g;g)\, h^i, \quad B^{\mathrm{r}}(g;g) = \sum_{i=0}^{\infty} B_i^{\mathrm{r}}(g;g)\, h^i.$$

We write the inverse of the formal power series $B(x;y)$ as

$$(B^{-1})(x;y) = \sum_{i=0}^{\infty} \hat{B}_i(x;y)\, h^i;$$

consequently

$$(B^{-1})^{\mathrm{r}}(g;g) = \sum_{i=0}^{\infty} \hat{B}_i^{\mathrm{r}}(g;g)\, h^i.$$

Lemma 6. *With notations as above and $\varphi, \psi \in C^{\infty}(\mathbf{G})$, we have*

$$\begin{aligned}&((B^{-1})^{\mathrm{r}} \circ B^{\ell})(g_1 \cdot g_2; g_1 \cdot g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2))\\ &\quad = (B^{-1})^{\mathrm{r}}(g_1; g_1)(B^{\ell})(g_2; g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2)).\end{aligned}$$

Proof. The left-hand member is equal to

$$\begin{aligned}&(B^{-1})^{\mathrm{r}}(g_1 \cdot g_2; g_1 \cdot g_2) \cdot B^{\ell}(g_1 \cdot g_2; g_1 \cdot g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2))\\ &\quad = (B^{-1})^{\mathrm{r}}(g_1 \cdot g_2; g_1 \cdot g_2) \cdot (B^{\ell}\, (\varphi \otimes \psi))(g_1 \cdot g_2; g_1 \cdot g_2)\\ &\quad = (B^{-1})^{\mathrm{r}}(g_1; g_1) \cdot (B^{\ell}(\varphi \otimes \psi))(g_1 \cdot g_2; g_1 \cdot g_2)\\ &\quad = (B^{-1})^{\mathrm{r}}(g_1; g_1) \cdot B^{\ell}(g_2; g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2)),\end{aligned}$$

where only the invariance properties of $(B^{-1})^{\mathrm{r}}$ and B^{ℓ} have been used. □

Lemma 7. *With notations as above, we have*

$$\begin{aligned}&(B^{-1})^{\mathrm{r}}(g_1; g_1) \cdot (B^{\ell})(g_1; g_1)\\ &\qquad \cdot (B^{-1})^{\mathrm{r}}(g_2; g_2) \cdot B^{\ell}(g_2; g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2))\\ &\quad = (B^{-1})^{\mathrm{r}}(g_1; g_1) \cdot (B^{\ell})(g_2; g_2)\, (\varphi(g_1 \cdot g_2) \otimes \psi(g_1 \cdot g_2)).\end{aligned}$$

Proof. Let χ be an element in $C^\infty(\mathbf{G}\times\mathbf{G})$. We have

$$\begin{aligned} &B^\ell(g_1;g_1)\cdot(B^{-1})^{\mathrm{r}}(g_2;g_2)\cdot\chi(g_1\cdot g_2;g_1\cdot g_2)\\ &\quad= B^\ell(g_1;g_1)\,(B^{-1})^\ell(g_1;g_1)\,\chi(g_1\cdot g_2;g_1\cdot g_2)\\ &\quad= (B\circ B^{-1})^\ell\,(g_1;g_1) = \chi(g_1\cdot g_2;g_1\cdot g_2)\,, \end{aligned}$$

where lemma 5 has been used. □

From the latter two lemmae, we obtain

Lemma 8. *With notations as above*

$$\begin{aligned} &((B^{-1})^{\mathrm{r}}\circ B^\ell)(g_1\cdot g_2;g_1\cdot g_2)\,(\varphi(g_1\cdot g_2)\otimes\psi(g_1\cdot g_2)) \qquad (12)\\ &\quad= ((B^{-1})^{\mathrm{r}}\circ B^\ell)(g_1;g_1)\cdot((B^{-1})^{\mathrm{r}}\cdot B^\ell)(g_2;g_2)\,(\varphi(g_1\cdot g_2)\otimes\psi(g_1\cdot g_2))\,. \end{aligned}$$

We can write this result in terms of the coproduct

$$\Delta: C^\infty(\mathbf{G})\longrightarrow C^\infty(\mathbf{G})\,\hat{\otimes}\,C^\infty(\mathbf{G})\,,\quad \varphi\longrightarrow\Delta\varphi\,,$$

$$\Delta\varphi(g_1;g_2) = \varphi(g_1\cdot g_2)\,.$$

We have thus obtained

Theorem 9. *With notations as above, we have*

$$\Delta[((B^{-1})^{\mathrm{r}}\circ B^\ell)\,(\varphi\otimes\psi)] = ((B^{-1})^{\mathrm{r}}\circ B^\ell)\,(\Delta\varphi\otimes\Delta\psi)\,, \qquad (13)$$

where the right-hand side is, by definition, the right-hand side in lemma 8.

We will apply this theorem in the next section. When $B\equiv F$ is a star product the theorem will prove the fundamental property (3).

4. The invariant differential Hochschild cohomology

1. Let $\mathbf{M}$ be a C^∞ manifold. The Hochschild cohomology is defined as follows:

Definition 10. A p-cochain is a p-linear map

$$C: C^\infty(\mathbf{M})\times\overset{p}{\cdots}\times C^\infty(\mathbf{M})\longrightarrow C^\infty(\mathbf{M})\,,$$

$$(f_1;\ldots;f_p)\longrightarrow C(f_1;\ldots;f_p)\,,$$

defined by a differential operator on each argument with no constant term.

Let $C^p \equiv C^p(C^\infty(\mathbf{M}))$ be the vector space of these cochains.

Definition 11. The cohomology operator is defined by

$$\mathbf{d} : C^p \longrightarrow C^{p+1}, \quad C \longrightarrow \mathbf{d}C,$$

where

$$\begin{aligned}\mathbf{d}C(f_0; f_1; \dots; f_p) &= f_0 \cdot C(f_1; \dots; f_p) \\ &\quad + \sum_{i=1}^{\infty} (-1)^i C(f_0; f_1; \dots; f_{i-1} \cdot f_i; f_{i+1}; \dots; f_p) \\ &\quad + (-1)^{p+1} C(f_0; \dots; f_{p-1}) \cdot f_p .\end{aligned}$$

We then have $\mathbf{d} \circ \mathbf{d} = 0$.

Let $H^p(C^\infty(\mathbf{M})) \equiv H^p(C^\infty(\mathbf{M}); C^\infty(\mathbf{M}))$ be the pth space in this cohomology. We now have the following theorem:

Theorem 12 (J. Vey). *The space $H^p(C^\infty(\mathbf{M}))$ is isomorphic to the space of contravariant skew-symmetric p-tensors $\Lambda_p(\mathbf{M})$ on $\mathbf{M}$. This isomorphism is given by the splitting*

$$\alpha = \mathbf{A}\alpha + \mathbf{d}E, \quad \alpha \in C^p, \quad \mathbf{d}\alpha = 0, \quad E \in C^{p-1},$$

where $\mathbf{A}$ is the operator of complete skew symmetrization. In particular, the skew-symmetrized part of a p-cocycle is a contravariant (skew-symmetric) p-tensor.

2. Now let $\mathbf{M}$ be a Lie group $\mathbf{G}$. The (left- or right-) invariant Hochschild cohomology is defined as before, with the additional condition that the multi-differential operators C on $\mathbf{G}$ are (left or right) invariant. We can thus define the following cohomology, which by left or right translations is isomorphic to the left- or right-invariant Hochschild cohomology.

(i) The p-cochains are the elements of

$$\mathfrak{A}(\mathfrak{g}) \otimes \cdots \otimes \mathfrak{A}(\mathfrak{g}) .$$

(ii) The cohomology operator is

$$\mathbf{d} : \mathfrak{A}(\mathfrak{g})^{\otimes p} \longrightarrow \mathfrak{A}(\mathfrak{g})^{\otimes(p+1)},$$

$$\begin{aligned}\mathbf{d}(u_1 \otimes \cdots \otimes u_p) &= 1 \otimes u_1 \otimes \cdots \otimes u_p \\ &\quad + \sum_{i=1}^{p} (-1)^i u_1 \otimes \cdots \otimes \mathbf{c}(u_i) \otimes \cdots \otimes u_p \\ &\quad + (-1)^{p+1} u_1 \otimes \cdots \otimes u_p \otimes 1 .\end{aligned}$$

Hence $\mathbf{d} \circ \mathbf{d} = 0$.

We now have the following

Theorem 13 (A. Lichnerowicz). *Let $p = 2, 3$. The pth space of the invariant Hochschild cohomology on* $\mathbf{G}$ *is isomorphic to the space of invariant skew-symmetric p-tensors on* $\mathbf{G}$. *This isomorphism is given by the splitting*

$$\alpha = \mathbf{A}\alpha + \mathbf{d}E,$$

where α is an invariant p-cocycle, and where E is some invariant $(p-1)$-cochain.

Concerning the isomorphism in this theorem, see also ref. [27], pp. 5-15 to 5-18.

5. Star products on G

1.

Definition 14. A (left- or right-) invariant star product on $\mathbf{G}$ is a bilinear mapping on $C^\infty(\mathbf{G})[[h]]$ with values in this space defined in the following way:

(i) If φ, $\psi \in C^\infty(\mathbf{G})$,

$$\varphi * \psi = \sum_{i=1}^{\infty} C_i(\varphi; \psi)\, h^i, \quad C_0 = I,$$

$*$ is linearly defined on $C^\infty(\mathbf{G})[[h]]$.

(ii) C_i is an invariant bidifferential operator on $\mathbf{G}$ such that

$$C_i(\varphi; 1) = C_i(1; \varphi) = 0, \quad \forall \varphi \in C^\infty(\mathbf{G}).$$

(iii)

$$(\varphi * \psi) * \chi = \varphi * (\psi * \chi).$$

If C_i is left invariant, there is a unique $F_i \in \mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})$ such that

$$C_i(\varphi; \psi) = (F_i(x; y))^\ell\, (\varphi \otimes \psi).$$

Similary if C_i is right invariant.

We then have

$$(\varphi * \psi) * \chi = \sum_{j=1}^{\infty} C_j(\varphi * \psi; \chi)\, h^j = \sum_{m=0}^{\infty} \left(\sum_{i+j=m} C_j(C_i(\varphi; \psi); \chi) \right) h^m,$$

$$\varphi * (\psi * \chi) = \sum_{m=0}^{\infty} \left(\sum_{i+j=m} C_j(\varphi; C_i(\psi; \chi)) \right) h^m.$$

If (iii) is satisfied, we must have ($\forall m = 1,2,3,\ldots$)

$$\sum_{i+j=m} C_j(C_i(\varphi;\psi);\chi) = \sum_{i+j=m} C_j(\varphi; C_i(\psi;\chi)). \quad (14)$$

But

$$C_i(\varphi;\psi) = (F_i(x;y))^\ell(\varphi\otimes\psi).$$

Relation (14) is then equivalent to

$$\sum_{i+j=m} (F_j(x+y;z)\,F_i(x;y))^\ell\,(\varphi\otimes\psi\otimes\chi)$$

$$= \sum_{i+j=m} (F_j(x;y+z)\,F_i(x;y))^\ell\,(\varphi\otimes\psi\otimes\chi),$$

and the following equality ensues:

$$\sum_{i+j=m} F_j(x+y;z)\,F_i(x;y) = \sum_{i+j=m} F_j(x;y+z)\,F_i(y;z) \quad (m = 1,2,3,\ldots).$$

Proposition 15. *There is a bijective map between left-invariant star products on* **G** *and the elements*

$$F(x;y) = 1 + \sum_{i=1}^{\infty} F_i(x;y)\,h^i \in \mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})[[h]]$$

satisfying relations (4):

$$F(x+y;z)\,F(x;y) = F(x;y+z)\,F(y;z).$$

In brief, $F(x;y)$ is then a left-invariant star product on **G**.

2. If, in definition 14, the star product is right invariant, C_i determines a unique element $H_i(x;y)\in\mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})$ such that

$$C_i(\varphi;\psi) = (H_i(x;y))^{\mathrm{r}}\,(\varphi\otimes\psi).$$

Similarly, we prove

Proposition 16. *There is a bijective map between right-invariant star products on* **G** *and the elements*

$$H(x;y) = 1 + \sum_{i=1}^{\infty} H_i(x;y)\,h^i \in \mathfrak{A}(\mathfrak{g})\otimes\mathfrak{A}(\mathfrak{g})[[h]]$$

satisfying the relations

$$H(x;y)\,H(x+y;z) = H(y;z)\,H(x;y+z). \quad (15)$$

Proposition 17. *Let $F(x;y)$ be a left-invariant star product on* **G**. *Let $H(x;y) = F^{-1}(x;y)$. Then $H(x;y)$ is a right-invariant star product on* **G**.

Proof. Let us take inverses in relation (4). Then refer to (15). Note that $F^{-1}(x+y;z) = (F(x+y;z))^{-1}$. □

Define, as Takhtajan [23] does,

$$\varphi \overset{\circ}{*} \psi = (F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell})(\varphi \otimes \psi). \tag{16}$$

We then have

$$\begin{aligned}
&(\varphi \overset{\circ}{*} \psi) \overset{\circ}{*} \chi \\
&\quad = [F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell}][(F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell}(\varphi \otimes \psi)) \otimes \chi] \\
&\quad = (F^{-1}(x+y;z)^{\mathrm{r}} F(x+y;z)^{\ell} F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell})(\varphi \otimes \psi \otimes \chi) \\
&\quad = (F^{-1}(x+y;z)^{\mathrm{r}} F^{-1}(x;y)^{\mathrm{r}} F(x+y;z)^{\ell} F(x;y)^{\ell}))(\varphi \otimes \psi \otimes \chi) \\
&\quad = (F^{-1}(x;y) F^{-1}(x+y;z))^{\mathrm{r}} (F(x+y;z) F(x;y)^{\ell} (\varphi \otimes \psi \otimes \chi) \\
&\quad = (F^{-1}(y;z) F^{-1}(x;y+z))^{\mathrm{r}} (F(x;y+z) F(y;z))^{\ell} (\varphi \otimes \psi \otimes \chi) \\
&\quad = (F^{-1}(x;y+z)^{\mathrm{r}} F(x;y+z)^{\ell} F^{-1}(y;z)^{\mathrm{r}} F(y;z)^{\ell} (\varphi \otimes \psi \otimes \chi) \\
&\quad = [F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell}][\varphi \otimes (F^{-1}(x;y)^{\mathrm{r}} F^{-1}(x;y)^{\ell}(\psi \otimes \chi))] \\
&\quad = \varphi \overset{\circ}{*} (\psi \overset{\circ}{*} \chi).
\end{aligned}$$
□

3. Now, in lemma 8, let B be a star product F. Then the right-hand side of the equality in the lemma is $\Delta\varphi \overset{\circ}{*} \Delta\psi$. Hence, in this case, theorem 9 reads

$$\Delta(\varphi \overset{\circ}{*} \psi) = \Delta\varphi \overset{\circ}{*} \Delta\psi.$$

We have thus proved

Theorem 18. *Let $F(x;y)$ be a left-invariant star product on* **G**. *Then $F^{-1}(x;y)$ is a right-invariant star product on* **G** *and*

$$\varphi \overset{\circ}{*} \psi = (F^{-1}(x;y)^{\mathrm{r}} F(x;y)^{\ell})(\varphi \otimes \psi), \quad \varphi, \psi \in C^\infty(\mathbf{G}),$$

is a star product on **G**. *The coproduct*

$$\Delta : C^\infty(\mathbf{G}) \longrightarrow C^\infty(\mathbf{G}) \hat{\otimes} C^\infty(\mathbf{G}) \equiv C^\infty(\mathbf{G} \times \mathbf{G})$$

is a morphism of the non-commutative algebra $(C^\infty(\mathbf{G})[[h]], \overset{\circ}{*})$ to the non-commutative algebra $(C^\infty(\mathbf{G}) \hat{\otimes} C^\infty(\mathbf{G})[[h]]; \overset{\circ}{*})$.

Remark. Theorem 9 is true without reference to star products. This remark will prove very important when one extends this work to the quantization of quasi-triangular Poisson–Lie groups.

6. Invariant star products on G and the quantum Yang–Baxter equation

1. Let $F(x;y)$ be a left-invariant star product. Drinfeld considers the series [15]

$$S(x;y) = F^{-1}(y;x)\,F(x;y)$$

and states the following theorem, which we will prove below. (Also see ref. [16].)

Theorem 19. *The element $S(x;y)$ satisfies the QYBE (i′), (ii″).*

Proof. By hypothesis, we have

$$F(x+y;z)\,F(x;y) = F(x;y+z)\,F(y;z)\,. \tag{17}$$

We first remark that a similar relation holds for any permutation of (x,y,z). Clearly we have

$$F(x+y;z)\,F(y;x)\,S(x;y) = F(x;y+z)\,F(z;y)\,S(y;z)\,.$$

And from the above remark

$$F(y;x+z)\,F(x;z)\,S(x;y) = F(x+z;y)\,F(x;z)\,S(y;z)\,.$$

Again

$$F(y;x+z)\,F(z;x)\,S(x;z)\,S(x;y) = F(x+z;y)\,F(z;x)\,S(x;z)\,S(y;z)\,,$$

and in the same way

$$F(y+z;x)\,F(y;z)\,S(x;z)\,S(x;y) = F(z;x+y)\,F(x;y)\,S(x;z)\,S(y;z)\,.$$

Then

$$F(z+y;x)\,F(z;y)\,S(y;z)\,S(x;z)\,S(x;y)$$
$$= F(z;y+x)\,F(y;x)\,S(x;y)\,S(x;z)\,S(y;z)\,.$$

By using (17) again, we obtain relation (i′). □

7. The cohomological interpretation of the quantum Yang–Baxter equation

1. Let $F(x;y)$ be an invariant star product as in proposition 15. Relation (17) is equivalent to the set of relations $(m = 1,2,3,\dots)$

$$F_m(x+y;z) + F_m(x;y) - F_m(x;y+z) - F_m(y;z) = -\alpha_m(x;y;z)\,,$$

where

$$\alpha_m(x;y;z) = \sum_{\substack{i+j=m \\ i,j\geq 1}} [F_i(x+y;z)\,F_j(x;y) - F_i(x;y+z)\,F_j(y;z)]\,.$$

In terms of the complex $(\mathcal{T}\mathfrak{A}(\mathfrak{g});\mathbf{d})$, these relations are

$$\mathbf{d}F_m(x;y;z) = \alpha_m(x;y;z) \quad (m = 1,2,3,\ldots)\,.$$

Definition 20. Let $F(x;y)$ be now an arbitrary element in $\mathfrak{A}(\mathfrak{g})^{\otimes 2}[[h]]$,

$$F(x;y) = 1 + \sum_{i=1} F_i(x;y)h^i\,. \tag{18}$$

It defines an invariant star product to order $(m-1)$ if

$$\mathbf{d}F_i(x;y;z) = \alpha_i(x;y;z) \quad (i = 1,2,\ldots,m-1)\,.$$

Theorem 21 (Gerstenhaber). *If (18) defines a star product to order $(m-1)$, $\alpha_m(x;y;z)$ is a three-cocycle. This star product can be extended to order m if and only if this cocycle is exact.*

If we now refer to theorem 13, we have

Corollary 22. *If (18) defines a star product to order $(m-1)$, this star product extends to order m if and only if* $\mathrm{A}\alpha_m(x;y;z) = 0$.

2. Proof of theorem 2 and corollary 4. Let

$$F(x;y) = 1 + \sum_{i=1}^{\infty} F_i(x;y)\,h^i$$

be an arbitrary element of $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$. We consider the following expressions:

$$X(x;y;z) = S(x;y)\,S(x;z)\,S(y;z) - S(y;z)\,S(x;z)\,S(x;y)\,,$$

$$Y(x;y;z) = F(x+y;z)\,F(x;y) - F(x;y+z)\,F(y;z)\,, \tag{i}$$

$$F(x+y;z)\,F(x;y) = Y(x;y;z) + F(x;y+z)\,F(y;z)\,, \tag{ii}$$

$$F(x;y+z)\,F(y;z) = F(x+y;z)\,F(x;y) - Y(x;y;z)\,. \tag{iii}$$

From (i) we obtain

$$Y(x;y;z) = F(x+y;z)\,F(y;x)\,S(x;y) - F(x;y+z)\,F(z;y)\,S(y;z)\,, \tag{iv}$$

and from (ii) and (iii) we obtain

$$F(x+y;z)\,F(y;x) = Y(y;x;z) + F(y;x+z)\,F(x;z)\,, \qquad \text{(ii}'\text{)}$$

$$F(x;y+z)\,F(z;y) = F(x+z;y)\,F(x;z) - X(x;z;y)\,. \qquad \text{(iii}'\text{)}$$

Given (ii′) and (iii′), (iv) becomes

$$\begin{aligned} Y(x;y;z) =& Y(y;x;z)\,S(x;y) + Y(x;z;y)\,S(y;z) \\ &+ F(y;x+z)\,F(x;z)\,S(x;y) - F(x+z;y)\,F(x;z)\,S(y;z)\,. \end{aligned} \qquad \text{(v)}$$

If we now define

$$M(x;y;z) = Y(y;x;z)\,S(x;y) + Y(x;z;y)\,S(y;z)\,,$$

equality (v) becomes

$$\begin{aligned} Y(x;y;z) =& M(x;y;z) + F(y;x+z)\,F(z;x)\,S(x;z)\,S(x;y) \\ &- F(x+z;y)\,F(z;x)\,S(x;z)\,S(y;z)\,. \end{aligned} \qquad \text{(vi)}$$

But from (ii) and (iii),

$$F(z+x;y)\,F(z;x) = Y(z;x;y) + F(z;x+y)\,F(x;y)\,, \qquad \text{(ii}''\text{)}$$

$$F(y;z+x)\,F(z;x) = F(y+z;x)\,F(y;z) - Y(y;z;x)\,, \qquad \text{(iii}''\text{)}$$

and from (vi),

$$Y(x;y;z) = M(x;y;z) - N(x;y;z) + P(x;y;z) - Q(x;y;z)\,, \qquad \text{(vii)}$$

where

$$N(x;y;z) = Y(y;z;x)\,S(x;z)\,S(x;y) + Y(z;x;y)\,S(x;z)\,S(y;z)\,,$$

$$P(x;y;z) = F(y+z;x)\,F(z;y)\,S(y;z)\,S(x;z)\,S(x;y)\,,$$

$$Q(x;y;z) = F(z;x+y)\,F(y;x)\,S(x;y)\,S(x;z)\,S(y;z)\,.$$

By separately considering the terms of the formal power series in the identity (vii), we see that

- There is no term in h^0.
- The term in h^1 is

$$\begin{aligned} Y_1(x;y;z) =& M_1(x;y;z) - N_1(x;y;z) \\ &+ F_1(y+z;x) + F_1(z;y) + S_1(y;z) + S_1(x;z) + S_1(x;y) \\ &- F_1(z;x+y) - F_1(y;x) - S_1(x;y) - S_1(x;z) - S_1(y;z)\,. \end{aligned}$$

That is to say,

$$Y_1(x;y;z) = M_1(x;y;z) - N_1(x;y;z) + F_1(y+z;x) - F_1(z;x+y) + F_1(z;y) - F_1(y;x).$$

But

$$M_1(x;y;z) = Y_1(y;x;z) + Y_1(x;z;y),$$

$$N_1(x;y;z) = Y_1(y;z;x) + Y_1(z;x;y),$$

$$Y_1(z;y;x) = F_1(y+z;x) + F_1(z;y) - F_1(z;x+y) - F_1(y;x);$$

thus, we obtain

$$Y_1(x;y;z) - Y_1(y;x;z) - Y_1(x;z;y) + Y_1(y;z;x) + Y_1(z;x;y) - Y_1(z;y;x) = 0.$$

That is to say,

$$\mathbf{A}Y_1(x;y;z) = 0.$$

Thus, the above relation is satisfied for any $F(x;y)$. This is an interesting triviality. In fact, from the definition of $Y(x;y;z)$, we have

$$Y_1(x;y;z) = F_1(x;y;z) + F_1(x;y) - F_1(x;y+z) - F_1(y;z) = \mathbf{d}F_1(x;y;z).$$

$Y_1(x;y;z)$ is then an exact cocycle; $\mathbf{A}Y_1(x;y;z) = 0$ must be satisfied according to theorems 12, 13.

– The term in h^2 is

$$Y_2(x;y;z) = M_2(x;y;z) - N_2(x;y;z) + P_2(x;y;z) - Q_2(x;y;z),$$

where

$$P_2(x;y;z) = [F(y+z;x)\,F(z;y)\,S(y;z)\,S(x;z)\,S(x;y)]_2,$$

$$Q_2(x;y:z) = [F(z;y+x)\,F(y;x)\,S(x;y)\,S(x;z)\,S(y;z)]_2.$$

Then

$$M_2(x;y;z) = Y_1(y;x;z)\,S_1(x;y) + Y_2(y;x;z) + Y_1(x;y;z)\,S_1(y;z) + Y_2(x;z;y),$$

$$N_2(x;y;z) = Y_2(y;z;x) + Y_1(y;z;x)\,[S_1(x;z) + S_1(x;y)] + Y_2(z;x;y) + Y_1(z;x;y)\,[S_1(x;z) + S_1(y;z)],$$

$$P_2(x;y;z) = [F(z+y;x)F(z;y)]_2 + [S(y;z)S(x;z)S(x;y)]_2 + [F(z+y;x)F(z;y)]_1 \cdot [S(y;z)S(x;z)S(x;y)]_1,$$

$$Q_2(x;y;z) = [F(z;x+y)F(y;z)]_2 + [S(x;y)S(x;z)S(y;z)]_2 + [F(z;x+y)F(y;x)]_1 \cdot [S(x;y)S(x;z)S(y;z)]_1.$$

And hence

$$\begin{aligned} Y_2(x;y;z) = {} & Y_1(y;x;z)S_1(x;y) + Y_2(y;x;z) \\ & + Y_1(x;z;y)S_1(y;z) + Y_2(x;z;y) \\ & - Y_1(y;z;x)[S_1(x;z) + S_1(x;y)] \\ & - Y_1(z;x;y)[S_1(x;z) + S_1(x;y)] \\ & - Y_2(y;z;x) - Y_2(z;x;y) + Y_2(z;y;x) \\ & + Y_1(z;y;x)[S(y;x)S(x;z)S(x;y)]_1 - X_2(x;y;z). \end{aligned}$$

But if now we suppose $F(x;y)$ to be a star product to order 1, definition 20, we have

$$Y_1(x;y;z) = -\mathbf{d}F_1(x;y;z) = 0,$$

and so, by this hypothesis,

$$Y_2(x;y;z) - Y_2(y;x;z) - Y_2(x;z;y) + Y_2(y;z;x) + Y_2(z;x;y) + Y_2(z;y;x) = -X_2(x;y;z).$$

That is to say,

$$\begin{aligned} 6\mathbf{A}Y_2(x;y;z) &= -[S(x;y)S(x;z)S(y;z) - S(y;z)S(x;z)S(x;y)]_2 \\ &= -[S_1;S_1]. \end{aligned}$$

But

$$Y_2(x;y;z) = F_2(x+y;z) + F_2(x;y) - F_2(x;y+z) - F_2(y;z) + [F(x+y;z)F(x;y) - F(x;y+z)F(y;z)]_1,$$

and hence

$$Y_2(x;y;z) = -\mathbf{d}F_2(x;y;z) + \alpha_2(x;y;z).$$

From this we obtain

$$6\mathbf{A}\alpha_2(x;y;z) = -[S(x;y)S(x;z)S(y;z) - S(y;z)S(x;z)S(x;y)]_2.$$

We can then assert

A star product to order 1 (i.e., $\mathbf{d}F_1(x;y;z) = Y_1(x;y;z) = 0$*) can be extended to order 2 (i.e.,* $\exists F_2 \mid \mathbf{d}F_2(x;y;z) = \alpha_2(x;y;z)$*) if and only if the classical Yang–Baxter equation* $[S_1;S_1] = 0$ $(S_1(x;y) = F_1(x;y) - F_1(y;x))$ *is satisfied (i.e., the QYBE is satisfied to order 2,* $X_2(x;y;z) = 0$*).*

This result can be generalized by induction to any order.

– The term in h^k, for any k, is

$$Y_k(x;y;z) = M_k(x;y;z) - N_k(x;y;z) + P_k(x;y;z) - Q_k(x;y;z).$$

But

$$M_k(x;y;z) = Y_k(y;x;z) + \sum_{i+j=k} Y_i(y;x;z)\,S_j(x;y)$$

$$+ Y_k(x;z;y) + \sum_{i+j=k} Y_i(x;z;y)\,S_j(y;z) \quad (j>0).$$

If we now suppose that the QYBE is satisfied to order k [that is, if $F(x;y)$ is a star product to order $(k-1)$], we have

$$Y_i(x;y;z) = 0 \quad (i = 1,2,\dots,k-1),$$

and thus

$$M_k(x;y;z) = Y_k(y;x;z) + Y_k(x;z;y) \ ,$$

$$N_k(x;y;z) = Y_k(y;z;x) + Y_k(z;x;y) \ .$$

On the other hand,

$$P_k(x;y;z) - Q_k(x;y;z)$$

$$= [F(y+z;x)\,F(z;y) - F(z;x+y)\,F(y;x)]_k$$

$$+ \sum_{i+j=k} [F(y+z;x)\,F(x;y)]_i\,[S(y;z)\,S(x;z)\,S(x;y)]_j$$

$$- \sum_{i+j=k} [F(z;x+y)\,F(y;x)]_i\,[S(x;y)\,S(x;z)\,S(y;z)]_j$$

$$= Y_k(z;y;x) - X_k(x;y;z)$$

$$- \sum_{\substack{i+j=k\\ j\geq 1}} [F(z;y+z)\,F(y;x)]_i\,X_j(x;y;z).$$

From this we obtain

$$Y_k(x;y;z) = Y_k(y;x;z) + Y_k(x;z;y)$$

$$- Y_k(y;z;x) - Y_k(z;x;y) + Y_k(z;y;x) - X_k(x;y;z).$$

That is to say,

$$6\mathbf{A}Y_k(x;y;z) = -X_k(x;y;z).$$

But

$$Y_k(x;y;z) = -\mathbf{d}F_k(x;y;z) + \alpha_k(x;y;z);$$

hence

$$6\mathbf{A}\alpha_k(x;y;z) = -X_k(x;y;z).$$

Consequently, a star product $F(x;y)$ to order $(k-1)$ can be extended to a star product to order k if and only if the QYBE is satisfied to order k.

If we refer to theorem 13, the proof of theorem 2 and corollary 4 is complete.□

8. A converse of the foregoing theorem

In this section we prove part 1 of theorem 1. That is to say,

Theorem 23. *Let $R \in \mathrm{End}(\mathbb{R}^n \otimes \mathbb{R}^n)[[h]]$ be such that the equations*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}, \tag{i}$$

$$R^{12}R^{21} = I, \tag{ii}$$

are satisfied. Then there is a star product $F(x;y)$ on the group $\mathrm{GL}(n;\mathbb{R})$ such that if $S(x;y) = F^{-1}(y;x)F(x;y)$ we have

$$(\mathbf{P} \otimes \mathbf{P})S(x;y) = R,$$

where

$$\mathbf{P} : \mathfrak{gl}(n;\mathbb{R}) \longrightarrow \mathrm{End}(\mathbb{R}^n)$$

is the natural representation of the Lie algebra $\mathfrak{gl}(n;\mathbb{R})$.

First two lemmae.

Lemma 24. *Let*

$$F = 1 + \sum_{i=1}^{\infty} F_i h^i$$

be an element of $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$. We write

$$F^{-1} = 1 + \sum_{i=1}^{\infty} \hat{F}_i h^i, \qquad S(x;y) = F^{-1}(y;x)F(x;y).$$

Then

$$\hat{F}_0(x;y) = 1, \qquad \hat{F}_1(x;y) = -F_1(x;y), \tag{1}$$

$$\hat{F}_r(x;y) = -F_r(x;y) - \sum_{\substack{l+k=r \\ l\geq 1;k\geq 1}} \hat{F}_l(x;y)\,F_k(x;y)\,, \tag{2}$$

$$S_1(x;y) = F_1(x;y) - F_1(y;x)\,, \tag{3}$$

$$\begin{aligned} S_r(x;y) =& F_r(x;y) - F_r(y;x) \\ &+ \sum_{\substack{i+j=r \\ i\geq 1;j\geq 1}} \hat{F}_i(y;x)\,(F_j(x;y) - F_j(y;x)) \quad (r = 2,3,\dots)\,. \end{aligned} \tag{4}$$

We rewrite (4) as

$$S_r(x;y) = F_r(x;y) - F_r(y;x) + R_r(F_1,\dots,F_{r-1})(x;y)\,. \tag{5}$$

Proof. By straightforward calculations on

$$F^{-1}(x;y)\,F(x;y) = 1\,, \qquad S(x;y) = F^{-1}(y;x)\,F(x;y)\,. \qquad \square$$

Remark. For our purpose it is important to remark that R_r depends only on $F_1,\dots,F_{r-1}$, because $\hat{F}_i$ depends only on F_l with $1 \leq l \leq i$.

Lemma 25. *Let F and F' be any two elements in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$. Then*
(i) for $r = 1,2,3,\dots$,

$$\begin{aligned} S'_r(x;y) - S_r(x;y) =& [F'_r(x;y) - F_r(x;y)] - [F'_r(y;x) - F_r(y;x)] \\ &+ R_r(F'_1,\dots,F'_{r-1})(x;y) - R_r(F_1,\dots,F_r)(x;y)\,; \end{aligned}$$

(ii) if

$$F_i(x;y) = F'_i(x;y)\,, \quad i = 1,\dots,r-1\,,$$

then $S'_r(x;y) - S_r(x;y)$ is skew symmetric, and in the case $\mathfrak{g} \equiv \mathfrak{gl}(n;\mathbb{R})$,

$$(\mathbf{P}\otimes\mathbf{P})\,[S'_r(x;y) - S_r(x;y)] \in \mathfrak{gl}(n;\mathbb{R}) \otimes \mathfrak{gl}(n;\mathbb{R})$$

is a Hochschild two-cocycle on the group $\mathrm{GL}(n;\mathbb{R})$.

Proof. By a straightforward calculation from (5) in lemma 24. $\square$

Proof of theorem 23. Let $F(x;y)$ be the element to be found. In $\mathfrak{A}(\mathfrak{gl}(n;\mathbb{R}))$ we must have

$$S_1^{12}S_1^{13} + S_1^{12}S_1^{23} + S_1^{13}S_1^{23} - S_1^{13}S_1^{12} - S_1^{23}S_1^{12} - S_1^{23}S_1^{13} = 0\,, \tag{a}$$

$$S_1^{12} + S_1^{21} = 0\,, \tag{b}$$

where

$$S_1^{12} \equiv S_1(x;y) = F_1(x;y) - F_1(y;x) \in \mathfrak{gl}(n;\mathbb{R}) \otimes \mathfrak{gl}(n;\mathbb{R})\,.$$

If we choose $S_1^{12} = r_1$, we obtain from (i)

$$[S_1^{12};S_1^{13}] + [S_1^{12};S_1^{23}] + [S_1^{13};S_1^{23}] = 0,$$

where the brackets are calculated in $\mathfrak{gl}(n;\mathbb{R}) \equiv \mathrm{End}(\mathbb{R}^n)$. But this expression can be written as in (a). Clearly, (b) is satisfied. Thus $S_1(x;y) = r_1$ is skew symmetric, and therefore a Hochschild two-cocycle. We can then define $F_1(x;y)$ as

$$F_1(x;y) = \tfrac{1}{2}S_1(x;y) = \tfrac{1}{2}r_1 .$$

$F_1(x;y)$ is determined in this way by the R given in the theorem.

We now proceed by induction. The hypothesis is as follows. Let

$$F(x;y) = 1 + \sum_{i=1}^{k-1} F_i(x;y)\, h^i$$

be a star product to order $(k-1)$, and define

$$S(x;y) = F^{-1}(y;x)\, F(x;y) .$$

We assume that the QYBE is satisfied to order k, eq. (11), and also that

$$(\mathbf{P} \otimes \mathbf{P})\, S_i(x;y) = r_i \quad (i = 1,2,\ldots,k-1),$$

where r_i is given in the hypothesis of the theorem. We must prove that there is an $F_k(x;y)$ such that

$$T(x;y) = 1 + S_1(x;y)\, h + \cdots + S_{k-1}(x;y)\, h^{k-1} + S_k(x;y)\, h^k$$

satisfies the QYBE to order $(k+1)$. Of course $S(x;y)$ and $T(x;y)$ coincide to order $(k-1)$. In fact, the equation

$$\mathbf{d}F_k(x;y;z) = \alpha_k(x;y;z)$$

has solutions, because $\mathrm{A}\alpha_k(x;y;z)$ is the QYBE to order k, which is satisfied by hypothesis. If we now take any solution $\overline{F}_k(x;y)$ of this equation, any other solution will have the form

$$F_k = \overline{F}_k + \beta_k + \mathbf{d}E_k ,$$

where $\beta_k \in \mathfrak{gl}(n;\mathbb{R}) \otimes \mathfrak{gl}(n;\mathbb{R})$ is any skew-symmetric Hochschild two-cocycle, and E_k is any one-cochain. From lemma 25

$$S_k - \overline{S}_k = 2\beta_k ;$$

hence

$$(\mathbf{P} \otimes \mathbf{P})(S_k - \overline{S}_k) = 2\beta_k = S_k - \overline{S}_k .$$

From $\overline{S}(x;y)\,\overline{S}(y;x) = 1$ we obtain

$$\overline{S}_k(x;y) + \overline{S}_k(y;x) + \sum_{\substack{i+j=k\\ i,j>0}} S_i(x;y)\, S_j(y;x) = 0 ;$$

on the other hand,

$$r_k^{12} + r_k^{21} + \sum_{\substack{i+j=k \\ i,j>0}} r_i^{12} r_j^{21} = 0;$$

hence, by induction hypothesis, we obtain

$$(\mathbf{P} \otimes \mathbf{P})\,(\overline{S}_k(x;y) + \overline{S}_k(y;x)) = r_k^{12} + r_k^{21},$$

which we can write as

$$r_k^{12} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(x;y) = -[r_k^{21} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(y;x)].$$

Then $r_k^{12} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(x;y)$ is in $\mathfrak{gl}(n;\mathbf{R})^{\otimes 2}$ and is skew symmetric. It is then a Hochschild two-cocycle. We take β_k as the value

$$2\beta_k = r_k^{12} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(x;y).$$

We now consider the solution F_k, where we set $E_k = 0$,

$$F_k(x;y) = \overline{F}_k(x;y) + \tfrac{1}{2}(r_k^{12} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(x;y)).$$

From this we obtain

$$S_k(x;y) = \overline{S}_k(x;y) + (r_k^{12} - (\mathbf{P} \otimes \mathbf{P})\,\overline{S}_k(x;y)),$$

where, clearly,

$$(\mathbf{P} \otimes \mathbf{P})\,S_k(x;y) = r_k^{12}.$$

On the other hand,

$$\mathrm{A}\alpha_{k+1}(x;y;z) = -\tfrac{1}{6}[S(x;y)\,S(x;z)\,S(y;z) - S(y;z)\,S(x;z)\,S(x;y)]_{k+1}$$

is a Hochschild three-cocycle (theorems 2, 13). Then

$$\begin{aligned} \mathrm{A}\alpha_{k+1}(x;y;z) &= (\mathbf{P} \otimes \mathbf{P} \otimes \mathbf{P})\,\mathrm{A}\alpha_{k+1}(x;y;z) \\ &= -\tfrac{1}{6}(R^{12}R^{13}R^{23} - R^{23}R^{13}R^{12})_{k+1} = 0. \end{aligned}$$

The proof of the theorem is now complete. □

To prove part (2) of theorem 1, we need some basic facts about the equivalence of star products and a few preliminary properties.

9. Equivalence of invariant star products on G

1.

Definition 26. Let

$$F(x;y) = 1 + \sum_{i=1}^{\infty} F_i(x;y)\,h^i,$$

$$F'(x;y) = 1 + \sum_{j=1}^{\infty} F'_j(x;y)\, h^j ,$$

be any two elements in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$. We will say they are equivalent if there is some element

$$E = 1 + \sum_{i=1}^{\infty} E_i\, h^i \in \mathfrak{A}(\mathfrak{g})[[h]] \tag{19}$$

such that

$$E(x+y)\, F'(x;y) = F(x;y)\, E(x)\, E(y) .$$

Expanding the latter expression, we obtain

Proposition 27. *Elements $F(x;y)$ and $F'(x;y)$ are equivalent if and only if*

$$F'_k(x;y) - F_k(x;y) + G_k(x;y) = \mathbf{d}E_k(x;y) \quad (k = 1,2,3,\dots),$$

$$G_1(x;y) = 0, \tag{20}$$

where

$$G_k(x;y) \equiv G_k(E_1,\dots,E_{k-1};F'_1,\dots,F'_{k-1};F_1,\dots,F_{k-1})(x;y)$$

$$\equiv \sum_{i+j=k} [E_i(x+y)\, F'_j(x;y) - F_i(x;y)\, E_j(y) - F_i(x;y)\, E_j(x)] \tag{21}$$

$$- \sum_{i+j=k} E_i(x)\, E_j(y) - \sum_{i+j+l=k} F_i(x;y)\, E_j(x)\, E_l(y) \quad (i,j,l \geq 1) .$$

We should remark that $G_k(x;y)$ is defined by means of E_j, F_j, F'_j with $1 \leq j \leq k-1$.

Definition 28. Let F, F' be two given elements in $\mathfrak{A}(\mathfrak{g})^{\otimes 2}[[h]]$, and suppose there are $E_1,\dots,E_m \in \mathfrak{A}(\mathfrak{g})$ such that (20) is satisfied with $k = 1,2,\dots,m$. We will then say that F and F' are equivalent to order m.

Proposition 29. *Suppose now that $F(x;y)$ is an invariant star product. Let*

$$E = 1 + \sum_{i=1}^{\infty} E_i\, h^i \tag{22}$$

be an arbitrary element in $\mathfrak{A}(\mathfrak{g})[[h]]$. Define $F'(x;y)$ by the relation

$$F'(x;y) = E^{-1}(x+y)\, F(x;y)\, E(x)\, E(y) .$$

Then $F'(x;y)$ is an invariant star product.

Proof. We have

$$F'(x+y;z) = E^{-1}(x+y+z)\,F(x+y;z)\,E(x+y)\,E(z)\,,$$

$$F'(x+y;z)\,F'(x;y)$$
$$= E^{-1}(x+y+z)\,F(x+y;z)\,F(x;y)\,E(x)\,E(y)\,E(z)\,.$$

In the same way, we obtain

$$F'(x;y+z)\,F'(y;z)$$
$$= E^{-1}(x+y+z)\,F(x;y+z)\,F(y;z)\,E(x)\,E(y)\,E(z)\,,$$

but $F(x;y)$ satisfies (4), and hence $F'(x;y)$ as well. □

In view of this proposition the notion of equivalence of star products becomes meaningful.

Definition 30. (i) Two star products F and F' are equivalent if they are equivalent elements in the sense of definition 26.

(ii) They are equivalent to order m, if F and F' are equivalent elements to order m in the sense of definition 28.

2. Let F and F' be two star products. Suppose they are equivalent to order k, that is to say $(i = 1, \dots, k)$

$$F'_i - F_i + G_i(E_1, \dots, E_{i-1}; F'_i, \dots, F'_{i-1}; F_1, \dots, F_{i-1}) = \mathbf{d}E_i\,.$$

We know from refs. [9,11] that the two-cochain

$$F'_{k+1} - F_{k+1} + G_{k+1}(E_1, \dots, E_k; F'_1, \dots, F'_k; F_1, \dots, F_k)$$

is a two-cocycle.

From theorems 12, 13, there are $h_{k+1} \in \bigwedge^2(\mathfrak{g})$ and $E_{k+1} \in \mathfrak{A}(\mathfrak{g})$ such that

$$F'_{k+1} - F_{k+1} + G_{k+1}(E_1, \dots E_k; F'_1, \dots, F'_k; F_1, \dots, F_k)$$
$$= h_{k+1} + \mathbf{d}E_{k+1}\,. \qquad (23)$$

We have

Proposition 31. *Two invariant star products F and F' which are equivalent to order k are equivalent to order $k+1$ if and only if $h_{k+1} = 0$ in expression (23).*

10. A few preliminary lemmae

Lemma 32. *Let F and $\overline{F}$ be two elements in $\mathfrak{A}(\mathfrak{g}) \otimes \mathfrak{A}(\mathfrak{g})[[h]]$ such that*

$$E(x+y)\,\overline{F}(x;y) = F(x;y)\,E(x)\,E(y),$$

where E is as in (22). Define

$$S(x;y) = F^{-1}(y;x)\,F(x;y), \qquad \overline{S}(x;y) = \overline{F}^{-1}(y;x)\,\overline{F}(x;y).$$

We then have

(i)

$$\overline{S}(x;y) = E^{-1}(x)\,E^{-1}(y)\,S(x;y)\,E(x)\,E(y),$$

$$E^{-1}(x) = 1 + \sum_{i=1}^{\infty} \hat{E}_i h^i \quad (E E^{-1} = E^{-1} E = 1);$$

(ii) for $r = 1, 2, 3, \ldots,$

$$\overline{S}_r(x;y) = \hat{E}_r(x) + \hat{E}_r(y) + S_r(x;y) + E_r(x) + E_r(y)$$
$$+ \sum_{\substack{i+j+k+l+s=r \\ i,j,k,l,s \neq r}} \hat{E}_i(x;y)\,\hat{E}_j(x;y)\,S_k(x;y)\,E_l(x)\,E_s(y);$$

(iii) we will write the expression in (ii) as

$$\overline{S}_r = S_r + B_r(E_1, \ldots, E_{r-1}; S_1, \ldots, S_{r-1}; \hat{E}_1, \ldots, \hat{E}_{r-1}).$$

Proof. We obtain these expressions by straightforward calculations from the definitions. In (iii), we have used

$$\hat{E}_r + E_r = -\sum_{l+k=r} \hat{E}_l E_k \quad (l \geq 1, k \geq 1). \qquad \square$$

Remark. Note that the term $B_r(\cdots)$ in (iii) is a sum of products of E_i and S_i $(1 \leq i \leq r-1)$. In each one of these products, there is at least one E_i $(1 \leq i \leq r-1)$, but they need not contain an S_i $(1 \leq i \leq r-1)$.

Lemma 33. *Let F, F' be as in lemma 25(i). We assume they are equivalent to order k. Then $(r = 1, 2, 3, \ldots, k)$*

$$S'_r(x;y) - S_r(x;y) = -[G_r(x;y) - G_r(y;x)]$$
$$+ R_r(F'_1, \ldots, F'_{r-1})(x;y) - R_r(F_1, \ldots, F_{r-1})(x;y) \qquad (24)$$

$(R_1 = G_1 = 0 \Rightarrow S'_1 = S_1)$.

Proof. From definition 28, we obtain

$$F_i' - F_i = G_i + \mathbf{d}E_i \quad (i = 1,\dots,k).$$

If we now refer to the expression in lemma 25(i), we obtain eq. (24). Recall that $G_i(\cdots)$ is defined by expression (21). □

Lemma 34. *Let F and F' be two star products which are equivalent to order k. That is to say, we have*

$$F_i' - F_i + G_i = \mathbf{d}E_i \quad (i = 1,\dots,k-1), \tag{a}$$

$$F_{k+1}' - F_{k+1} + G_{k+1} = h_{k+1} + \mathbf{d}E_{k+1}, \tag{b}$$

where $h_{k+1} \in \bigwedge^2(\mathfrak{g})$. Then

$$S_{k+1}'(x;y) - S_{k+1}(x;y) = 2h_{k+1}(x;y) + A_{k+1}(\cdots)(x;y),$$

where

$$\begin{aligned} &A_{k+1}(F_1,\dots,F_k;F_1',\dots,F_k';E_1,\dots,E_k)(x;y) \\ &\quad = -[G_{k+1}(\cdots)(x;y) - G_{k+1}(\cdots)(y;x) \\ &\qquad + R_{k+1}(F_1',\dots,F_k')(x;y) - R_{k+1}(F_1,\dots,F_k)(x;y). \end{aligned}$$

Proof. We substitute (a) and (b) in (24), and we write $r = k+1$ in lemma 25(i). □

Lemma 35. *Let F and F' be two star products which are equivalent to order k. Let the element*

$$E = 1 + E_1 h + \cdots + E_k h^k \in \mathfrak{A}(\mathfrak{g})[[h]]$$

be responsible for the equivalence. Consider the star product $\overline{F}$, equivalent to F, defined by

$$\overline{F}(x;y) = E^{-1}(x+y)\,F(x;y)\,E(x)\,E(y).$$

Let S and $\overline{S}$ be elements defined in lemma 32. We then have

$$F_i' = \overline{F}_i \quad (i = 1,2,\dots,k), \tag{a}$$

$$\overline{S}_{k+1}(x;y) - S_{k+1}(x;y) = B_{k+1}(\hat{E}_1,\dots,\hat{E}_k;S_1,\dots,S_k;E_1,\dots,E_k)(x;y), \tag{b}$$

where

$$B_{k+1}(\cdots)(x;y) = \sum_{\substack{i+j+r+l+t=k+1\\ 0\le i,j,r,l,t\le k}} \hat{E}_i(x)\,\hat{E}_j(y)\,S_r(x;y)\,E_l(x)\,E_t(y),$$

$$\overline{S}_{k+1}(x;y) - S_{k+1}(x;y) = A_{k+1}(F_1,\dots,F_k;F'_1,\dots,F'_k;E_1,\dots,E_k), \quad \text{(c)}$$

where $A_{k+1}(\cdots)$ is defined in lemma 34.

Proof.

(a) Equivalence between F and F' and between F and $\overline{F}$ means ($i = 1,2,\dots,k$)

$$F'_i - F_i + G_i(F_1,\dots,F_{i-1};F'_1,\dots,F'_{i-1};E_1,\dots,E_{i-1} = \mathbf{d}E_i \quad ,$$

$$\overline{F}_i - F_i + G_i(F_1,\dots,F_{i-1};\overline{F}_1,\dots,\overline{F}_{i-1};E_1,\dots,E_{i-1}) = \mathbf{d}E_i \ .$$

Then, if $i = 1$,

$$F'_1 - F_1 = \mathbf{d}E_1\,, \qquad \overline{F}_1 - F_1 = \mathbf{d}E_1\,;$$

hence

$$F'_1 = \overline{F}_1\,.$$

If $i = 2$,

$$F'_2 - F_2 + G_2(F_1;F'_1;E_1) = \mathbf{d}E_2\,,$$

$$\overline{F}_2 - F_2 + G_2(F_1;F'_1;E_1) = \mathbf{d}E_2\,.$$

Hence

$$F'_2 = \overline{F}_2\,.$$

Similarly one deduces

$$F'_k = \overline{F}_k\,.$$

(b) We obtain this expression from (ii), (iii) in lemma 32, given that $E_i = 0\ ,\ i \geq k+1$.

(c) This expression is the same as in lemma 34, relative to the star products F, $\overline{F}$, which (being equivalent) are equivalent to order $(k+1)$. Hence $h_{k+1} = 0$, from proposition 31, and we can replace F_i by F'_i ($i = 1,\dots,k$), from (a). □

Theorem 36. *With notations as above,*

$$A_{k+1}(F_1,\dots,F_k;F'_1,\dots,F'_k;E_1,\dots,E_k)$$

$$= B_{k+1}(\hat{E}_1,\dots,\hat{E}_k;S_1,\dots,S_k;E_1,\dots,E_k).$$

Proof. By (b) and (c) of lemma 35. □

Lemma 37. *Given the Lie algebra* $\mathfrak{g} \equiv \mathfrak{gl}(n;\mathbb{R})$, *let*

$$\mathbf{P} : \mathfrak{gl}(n;\mathbb{R}) \longrightarrow \mathrm{End}(\mathbb{R}^n)$$

be the natural representation (identity), and (also!) let

$$\mathbf{P} : \mathfrak{A}(\mathfrak{gl}(n;\mathbb{R})) \longrightarrow \mathrm{End}(\mathbb{R}^n)$$

be the induced representation of its enveloping algebra. We have

(a) If $X \in \mathfrak{gl}(n;\mathbb{R})$, *then* $\mathbf{d}X = 0$.

(b) If $Y \in \mathfrak{A}(\mathfrak{gl}(n;\mathbb{R}))$, *there exists* $E \in \mathfrak{A}(\mathfrak{gl}(n;\mathbb{R}))$ *such that* $\mathbf{P}E = 0$ *and* $\mathbf{d}Y = \mathbf{d}E$.

Proof.

(a) Obvious. True even if $X \in \mathfrak{g}^{\otimes n}$.

(b) $\mathbf{P}Y \in \mathfrak{gl}(n;\mathbb{R}) \Rightarrow \mathbf{dP}Y = 0$ and

$$\mathbf{d}Y = \mathbf{d}Y - \mathbf{d}(\mathbf{P}Y) = \mathbf{d}(Y - \mathbf{P}Y) = \mathbf{d}E,$$

where $E = Y - \mathbf{P}Y$ but $\mathbf{P}E = \mathbf{P}Y - \mathbf{P}(\mathbf{P}Y) = \mathbf{P}Y - \mathbf{P}Y = 0$. □

Lemma 38. *In theorem 36, assume* $\mathfrak{g} \equiv \mathfrak{gl}(n;\mathbb{R})$ *and take* E_i *such that* $\mathbf{P}E_i = 0$, *allowed by lemma 37. We have*

$$(\mathbf{P} \otimes \mathbf{P}) A_{k+1}(F_1, \ldots, F_k; F'_1, \ldots, F'_k; E_1, \ldots, E_k) = 0.$$

Proof. From theorem 36, it suffices to prove

$$(\mathbf{P} \otimes \mathbf{P}) B_{k+1}(\hat{E}_1, \ldots, \hat{E}_k; F'_1, \ldots, F'_k; E_1, \ldots, E_k) = 0.$$

But this is true in view of the remark following lemma 32 and the choice of E_i, $\mathbf{P}E_i = 0$. □

11. Proof of part 2 in theorem 1

We are now ready to prove the following theorem (part 2 in theorem 1).

Theorem 39. *Let* F *be the star product constructed in theorem 23. Let* F' *be another star product satisfying the hypothesis of that theorem, i.e.,* $S'(x;y) = (F')^{-1}(y;x)\, F(x;y)$ *satisfies*

$$(\mathbf{P} \otimes \mathbf{P})\, S'(x;y) = R.$$

Then there exists

$$E = 1 + \sum_{i=1}^{\infty} E_i\, h^i,$$

where $E_i \in \mathfrak{A}(\mathfrak{gl}(n;\mathbb{R}))$ *and* $\mathbf{P}E_i = 0$, *such that*

$$F'(x;y) = E^{-1}(x+y)\, F(x;y)\, E(x)\, E(y).$$

Proof. From (3) of lemma 24, we have

$$S_1(x;y) = F_1(x;y) - F_1(y;x),$$
$$S_1'(x;y) = F_1'(x;y) - F_1'(y;x).$$

In theorem 23, we had

$$F_1(x;y) = \tfrac{1}{2}S_1(x;y);$$

but $\mathbf{d}F_1'(x;y) = 0$, hence

$$F_1'(x;y) = \tfrac{1}{2}S_1'(x;y) + \mathbf{d}E_1(x;y),$$

where $S_1'(x;y) \in \mathfrak{gl}(n;\mathbb{R})^{\otimes 2}$. By the hypothesis of the theorem on $S'(x;y)$,

$$(\mathbf{P}\otimes\mathbf{P})S_1(x;y) = S_1(x;y) = r_1 = S_1'(x;y) = (\mathbf{P}\otimes\mathbf{P})S_1'(x;y).$$

Hence

$$S_1(x;y) = r_1 = S_1'(x;y).$$

From this we obtain

$$F_1'(x;y) = F_1(x;y) + \mathbf{d}E_1(x;y),$$

where we have chosen $E_1(x)$ such that $\mathbf{P}E_1(x) = 0$, in accord with lemma 37. The star products F and F' are thus equivalent to order 1 (of course!).

We now proceed by induction. Suppose F and F' are equivalent to the order k, and we have chosen $\mathbf{P}E_i = 0$ $(i = 1,\dots,k)$. In consequence we have

$$F_{k+1}' - F_{k+1} + G_{k+1} = h_{k+1} + \mathbf{d}E_{k+1},$$

where $h_{k+1} \in \bigwedge^2(\mathfrak{gl}(n;\mathbb{R}))$ and $\mathbf{P}E_{k+1} = 0$. At this point, we allow for lemma 34. Thus

$$S_{k+1}' - S_{k+1} = 2h_{k+1} + A_{k+1}(F_1,\dots,F_k;F_1',\dots,F_k';E_1,\dots,E_k).$$

But

$$(\mathbf{P}\otimes\mathbf{P})A_{k+1} = 0$$

from lemma 38, and by hypothesis

$$(\mathbf{P}\otimes\mathbf{P})S_{k+1}'(x;y) = (\mathbf{P}\otimes\mathbf{P})S_{k+1}(x;y) = r_{k+1}.$$

Hence $h_{k+1} = 0$. By definition F and F' are equivalent to order $k+1$ (proposition 30), with $\mathbf{P}E_{k+1} = 0$. The proof of the theorem is now complete. □

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References

[1] V.G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang- Baxter equations, Sov. Math. Dokl. 27 (1983) 68–71.
[2] A. Weinstein, The local structure of Poisson manifolds, J. Diff. Geom. 18 (1983) 523–557.
[3] C. Moreno, Invariant star products and representations of compact semisimple Lie groups, Lett. Math. Phys. 12 (1986) 217.
[4] C. Moreno, Geodesic symmetries and invariant star products on Kähler symmetric spaces, Lett. Math. Phys. 13 (1987) 245–257.
[5] Y. Choquet-Bruhat and C. De Witt-Morette, *Analysis, Manifolds and Physics, Part II: 92 Applications* (North-Holland, Amsterdam, 1989).
[6] J.H. Lu and A. Weinstein, Poisson–Lie groups, dressing transformations and the Bruhat decomposition, J. Diff. Geom. 31 (1990) 501–526.
[7] Y. Kosmann-Schwarzbach and F. Magri, Poisson–Lie groups and complete integrability, Ann. Inst. H. Poincaré 49 (1990) 433–460.
[8] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantization. I and II, Ann. Phys. 111 (1978) 61–151.
[9] M. Gerstenhaber, Deformation theory of algebraic structures, Ann. Math. 79 (1964) 59–90.
[10] J. Vey, Déformations du crochet de Poisson sur une variété symplectique, Commun. Math. Helv. 50 (1975) 421–454.
[11] A. Lichnerowicz, Déformations d'algèbres associées à une variété symplectique (les $*_\nu$-produits), Ann. Inst. Fourier 32 (1982) 157–209.
[12] M. De Wilde and P. Lecompte, Existence of star-products and of formal deformations of the Poisson–Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. 7 (1983) 487.
[13] M. De Wilde and P. Lecompte, in: *Deformation Theory of Algebras and Structures and Applications*, eds. M. Hazewinkel and M. Gerstenhaber (Kluwer Acad. Publ., Dordrecht, 1988) pp. 897–960.
[14] P. Libermann and Ch. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).
[15] V.G. Drinfeld, On constant, quasiclassical solutions of the Yang–Baxter quantum equation, Sov. Math. Dokl. 28 (1983) 667–671.
[16] C. Moreno and L. Valero, Produits star sur certains G/K Kähleriens. Equation de Yang–Baxter et produits star sur G, in: Lecture Notes in Mathematics, Vol. 1416 (Springer, Berlin, 1990).
[17] C. Moreno and L. Valero, Cohomology de Hochschild de $C^\infty(\mathrm{G})$ et equation de Yang–Baxter quantique constante, in: Proc. Encuentros Españoles (Palma de Mallorca, 1990), pp. 231–255.
[18] C.Moreno and L. Valero, Produits star invariants et equation de Yang–Baxter quantique constante, in: Proc. Journées Relativistes (Grenoble, 1990) pp. 225–256.
[19] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Diff. Geom. 18 (1983) 523.
[20] C. Moreno and L. Valero, Quantum constant Yang–Baxter equation and Hochschild cohomology of $C^\infty(\mathrm{G})$, preprint (1989).
[21] C. Moreno and L. Valero, Invariant star products on a Lie group with an invariant Poisson structure, submitted.
[22] Y. Kosmann-Schwarzbach, Poisson–Drinfeld groups, in: *Non-linear Evolution Equations* (World Scientific, Singapore, 1987).
[23] L.A. Takhtajan, Introduction to quantum groups, preprint (1990).
[24] M. Rieffel, Deformation quantization of Heisenberg manifolds, Commun. Math. Phys. 122 (1989) 531–562.
[25] M. Rieffel, Lie groups, convolution algebras, deformation quantization of linear Poisson structures, Am. J. Math. 112 (1990) 657–686.
[26] S.L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys. 111 (1987) 613–615.
[27] Hyperalgèbres et groupes de Lie formels, Séminaire Sophus Lie, 2ème année 1955/56, Faculté des Sciences de Paris.